

NAVAL POSTGRADUATE SCHOOL

Monterey, California



ON DOMAIN DECOMPOSITION METHODS FOR
SOLVING PARTIAL DIFFERENTIAL EQUATIONS

BENY NETA
NAOTAKA OKAMOTO

MARCH 1990

Approved for public release; distribution unlimited
Prepared for:

Naval Postgraduate School
Monterey, CA 93943

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA 93943

Rear Admiral R. W. West, JR
Superintendent

Harrison Shull
Provost

This report was prepared in conjunction with research conducted for the Naval Postgraduate School and funded by the Naval Postgraduate School. Reproduction of all or part of this report is authorized.

Prepared by:

HAROLD M. FREDRICKSEN
Chairman
Department of Mathematics

G. E. SCHACHER
Dean of Faculty and
Graduate Education

DUDLEY KNOX LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY CA 93943-5101

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704 0188

a REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b RESTRICTIVE MARKINGS		
a SECURITY CLASSIFICATION AUTHORITY			3 DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited		
b DECLASSIFICATION/DOWNGRADING SCHEDULE					
PERFORMING ORGANIZATION REPORT NUMBER(S) NPS-53-90-004			5 MONITORING ORGANIZATION REPORT NUMBER(S) NPS-53-90-004		
a NAME OF PERFORMING ORGANIZATION Naval Postgraduate School		6b OFFICE SYMBOL (If applicable) MA	7a NAME OF MONITORING ORGANIZATION Naval Postgraduate School		
c ADDRESS (City, State, and ZIP Code) Monterey, Ca 93943			7b ADDRESS (City, State, and ZIP Code) Monterey, Ca 93943		
a NAME OF FUNDING / SPONSORING ORGANIZATION Naval Postgraduate School		8b OFFICE SYMBOL (If applicable) MA	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER O&MN, Direct funding		
c ADDRESS (City, State, and ZIP Code) Monterey, Ca 93943			10 SOURCE OF FUNDING NUMBERS		
			PROGRAM ELEMENT NO	PROJECT NO	TASK NO
			WORK UNIT ACCESSION NO		
1 TITLE (Include Security Classification) ON DOMAIN DECOMPOSITION METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATION					
2 PERSONAL AUTHOR(S) Beny Neta and Naotaka Okamoto					
3a TYPE OF REPORT technical report		13b TIME COVERED FROM 6/88 TO 3/90		14 DATE OF REPORT (Year, Month, Day) March 30, 1990	
15 PAGE COUNT 11					
6 SUPPLEMENTARY NOTATION					
7 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB-GROUP			
9 ABSTRACT (Continue on reverse if necessary and identify by block number) A domain decomposition method for solving partial differential equations is described. The conditions on interfaces will all be of Dirichlet type and obtained by the boundary element method using very few (less than 10) unknowns.					
0 DISTRIBUTION/AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS			21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
2a NAME OF RESPONSIBLE INDIVIDUAL Beny Neta			22b TELEPHONE (Include Area Code) (408) 646-2235		22c OFFICE SYMBOL MA

ON DOMAIN DECOMPOSITION METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

by

Beny Neta

Code MA Nd

Department of Mathematics

Naval Postgraduate School

Monterey, California 93943

and

Naotaka Okamoto

Okayama University of Science

Department of Applied Chemistry

Ridai-cho 1-1, Okayama 700

Japan

ABSTRACT

A domain decomposition method for solving partial differential equations is described. The conditions on interfaces will all be of Dirichlet type and obtained by the boundary element method using very few (less than 10) unknowns.

1. INTRODUCTION

Domain decomposition methods are based, as the name suggests, on subdivision of the domain into several subdomains and solving the problem on several subdomains in parallel. These methods are becoming the focus of research on numerical methods for the solution of partial differential equations (Widlund [34]). The methods can be regarded as divide-and-conquer algorithms since the problems on the subregions can be solved by well known techniques. The interactions between the solutions on the subdomains lead to an iterative procedure. When the number of subdomains is large one can improve the convergence of this iterative procedure by using a coarse grid to obtain starting values for the solution on interfaces. In this respect, the methods are similar to multigrid methods. The crucial point for domain decomposition schemes is how to pass information from one domain to other processors. Two different approaches were followed in the literature (see [2], [8], [10], [11], [16], [30], and references there). The first approach is based on decomposition of the domain into contiguous regions (see [13], [20]–[23], [25], [26], [33], and others). The second is based on having overlapping regions (Schwarz alternating method [9], [12], [18], [19], [28], [29], [32], [35], and others).

The main difficulty of such parallel techniques is in the initial assignment of values to the interfaces between domains. The more accurate such values are, the faster the convergence. As we mentioned earlier, one can use the solution on a coarse mesh (as in multigrid). Here we suggest the use of boundary element methods to approximate the solution at interfaces.

Boundary element methods were developed by Brebbia [5] and others. These methods are now widely used in various linear and nonlinear problems. Several papers ([1], [4], [6]–[8], [14], [17], [24], [25], [31]) are listed here. The list is by no means exhaustive.

In the next section, we introduce the problem and describe the domain decomposition method to be used. Section 3 will describe the boundary element method and its use for the approximation of interface values. In section 4, we give the details of the algorithm on Intel IPSC/2 Hypercube.

2. DOMAIN DECOMPOSITION

Consider the following elliptic problem

$$-\Delta u = f \quad \text{in} \quad \Omega \quad (1)$$

$$u = g \quad \text{on} \quad \partial\Omega \quad (2)$$

where Ω is the L shaped domain (Figure 1). The domain is divided into M subdomains Ω_i . The size of each subdomain will be such that the work is balanced among the processors. The subdomains are “colored” or numbered. Each subdomain borders subdomains of different colors (or numbers).

If we have boundary conditions for all M_1 subdomains numbered 1, one can assign these domains to the p processors for independent solution. Once the solution is obtained, the next set can be taken. If the domains overlap, there will be no need to transfer data; otherwise data is transferred to the neighboring subregions.

Remarks:

1. Boundary conditions on interfaces will be obtained following the recipe in the next section.
2. If $p < M_1$, then we solve for the first p subdomains numbered 1, transfer information if necessary and take the next p subdomains. If at some step one is left with less than

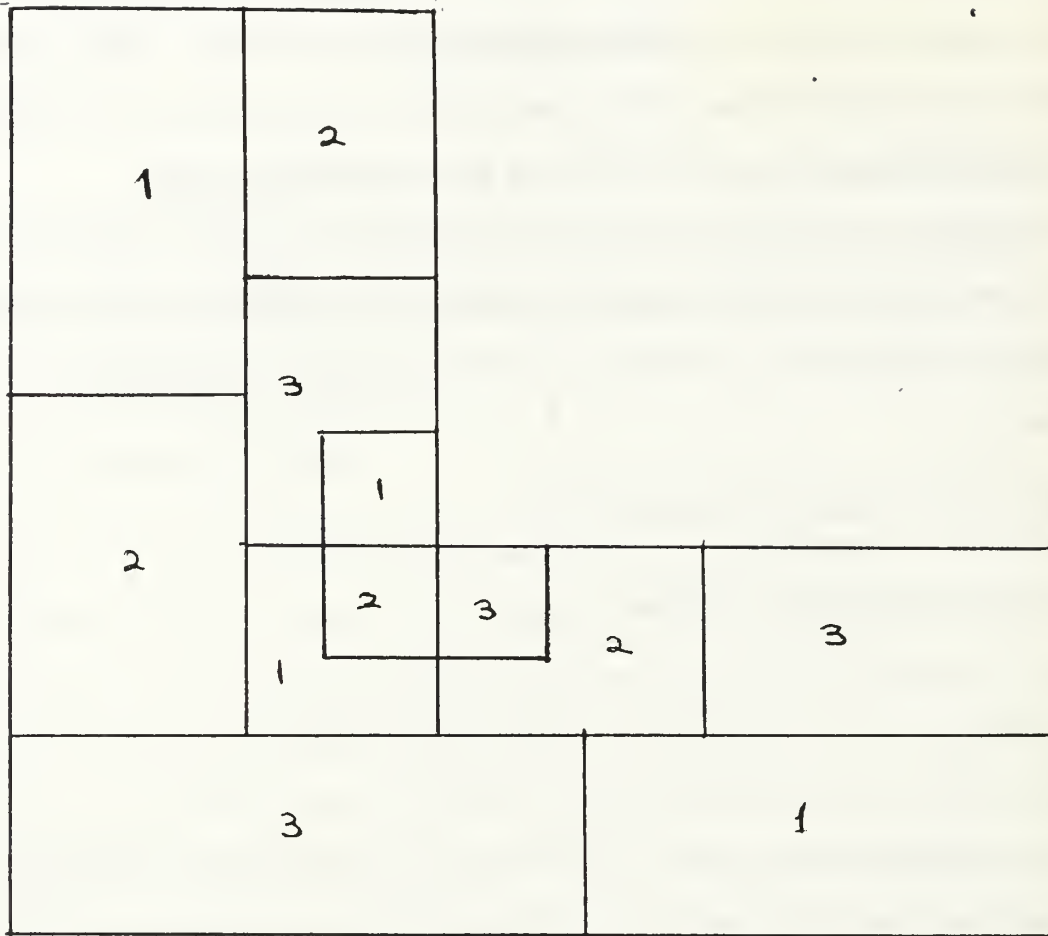


Figure 1:

p subregions of the same “color”, the other processors can start the next “color”.

3. BOUNDARY ELEMENT METHODS

“Boundary elements (Brebbia et al. [6]) have emerged as a powerful alternative to finite elements particularly in cases where better accuracy is required due to problems such as stress concentration or where the domain extends to infinity. The most important feature of boundary elements, however, is that it only requires discretization of the surface rather than the volume” (Brebbia and Dominguez [7]). Several examples were given there to the advantage of boundary elements.

The starting boundary integral equation required by the method can be deduced in a

simple way based, for example, on weighted residuals (see e.g., Brebbia and Dominguez [7]).

It was shown [7] that the problem

$$\nabla^2 u = 0 \quad \text{in} \quad \Omega \quad (3)$$

$$u = \bar{u} \quad \text{on} \quad \Gamma_D \quad (4)$$

$$q = \frac{\partial u}{\partial n} = \bar{q} \quad \text{on} \quad \Gamma_N \quad (5)$$

where n is the outward normal to the boundary $\Gamma = \Gamma_D \cup \Gamma_N$ is equivalent to

$$\int_{\Omega} (\nabla^2 u^*) u d\Omega = - \int_{\Gamma_N} \bar{q} u^* d\Gamma - \int_{\Gamma_D} q u^* d\Gamma + \int_{\Gamma_N} u q^* d\Gamma + \int_{\Gamma_D} \bar{u} q^* d\Gamma, \quad (6)$$

where u^* is a weight function with normal derivative q^* on the boundary. The boundary integral equation is

$$\frac{1}{2} u^i + \int_{\Gamma} u q^* d\Gamma = \int_{\Gamma} q u^* d\Gamma \quad (7)$$

where u^i is the value $u(x^i)$ and x^i is a boundary point. The numerical solution of the integral equation is accomplished by dividing Γ into N pieces Γ_i (Γ_D into N_1 pieces and Γ_N into N_2 pieces) and solving the system

$$\frac{1}{2} u^i + \sum_{j=1}^N \hat{H}^{ij} u^j = \sum_{j=1}^N G^{ij} q^j \quad i = 1, 2, \dots, N \quad (8)$$

where

$$\hat{H}^{ij} = \int_{\Gamma_j} q^* d\Gamma \quad (9)$$

$$G^{ij} = \int_{\Gamma_j} u^* d\Gamma. \quad (10)$$

In matrix form

$$HU = GQ \quad (11)$$

where

$$H^{ij} = \begin{cases} \hat{H}^{ij} & i \neq j \\ \hat{H}^{ij} + \frac{1}{2} & i = j \end{cases} \quad (12)$$

Note that N_1 values of u and N_2 values of q are known hence there are only N unknowns. One can rearrange the system so that X will contain the unknowns and solve

$$AX = F. \quad (13)$$

F is found by multiplying the corresponding columns by the known values of u 's or q 's. Once the boundary values are obtained, one can compute the interior values using

$$u^i = \int_{\Gamma} qu^* d\Gamma - \int_{\Gamma} uq^* d\Gamma, \quad (14)$$

or

$$u^i = \sum_{j=1}^N G^{ij} q^j - \sum_{j=1}^N \hat{H}^{ij} u^j. \quad (15)$$

In our case we use 7 points on the boundary and 1 interior point. The number of unknowns in this case is 9. One has to solve a system of 9 equations and then evaluate u at one point on each side of each subdomain Ω_i . This step can be done in parallel.

Remarks:

1. If the domain Ω was a rectangle, it is sufficient to take 4 points for the boundary element (one on each side).
2. The boundary element method was applied to inhomogeneous and nonlinear problems (see e.g., [17], [31]).
3. A list of fundamental solutions for various problems is given in Brebbia [8].

Acknowledgment

This research was conducted for the Office of Naval Research and was funded by the Naval Postgraduate School.

REFERENCES

1. Alarcon, E., Brebbia, C. A., and Dominguez, J., The boundary integral method in elasticity, *Intern. J. Mech. Sci.*, **20**, 1978, 625–639.
2. Bjørstad, P. E., Widlund, O. B., Solving elliptic problems on regions partitioned into substructures, in *Elliptic Problem Solvers III*, G. Birkhoff and A. Schoenstadt, eds., Academic Press, New York, 1984.
3. Bramble, J. H., Pasciak, J. E., and Schatz, A. H., An iterative method for elliptic problems on regions partitioned into substructures, *Math. Comp.*, **46**, 1986, 361–369.
4. Brebbia, C. A. and Dominguez, J., Boundary element methods for potential problems, *Applied Math. Modelling*, **1**, 7, Dec. 1977.
5. Brebbia, C. A., *The Boundary Element Method for Engineers*, Pentech Press, London, Computational Mechanics Publications, Boston, 1978.
6. Brebbia, C. A., Telles, J., and Wrobel, L., *Boundary Element Techniques — Theory and Applications in Engineering*, Springer-Verlag, Berlin and New York, 1984.
7. Brebbia, C. A., and Dominguez, J., *Boundary Elements An Introductory Course*, Computational Mechanics Publications, Boston, McGraw-Hill, New York, 1989.
8. Brebbia, C. A., *Progress in Boundary Element Methods*, Pentech Press, London.
9. Canuto, C. and Funaro, D., The Schwarz algorithm for spectral methods, *SIAM J. Numer. Anal.*, **25**, 1988, 24–40.
10. Chan, T. F., Glowinski, R., Periaux, J., and Widlund, O. B., *Domain Decomposition Methods*, SIAM, 1989.
11. Chan, T. F., Glowinski, R., and Widlund, O. B., *Third International Symposium on Domain Decomposition Methods*, SIAM, 1990.
12. Courant, R. and Hilbert, D., eds., *Methods of Mathematical Physics. Vol. 2*, Wiley-Interscience, New York, 1962.
13. Funaro, D., Multidomain spectral approximation of elliptic equations, *Numerical Methods for Partial Differential Equations 2*, 1986, 187–205.
14. Gipson, G. S., Boundary element fundamentals — Basic concepts and recent developments in the Poisson equation, in *Topics in Engineering Series, Vol. 2*, C. A. Brebbia and J. J. Connor, eds., Computational Mechanics Publications, Southampton, England, 1987.
15. Glowinski, R., Dinh, Q. V., and Periaux, J., Domain decomposition methods for non-linear problems in fluid dynamics, *Comput. Methods Appl. Mech. Engrg.*, **40**, 1983, 27–109.
16. Glowinski, R., Golub, G. H., Meurant, G. A., and Periaux, J., *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, SIAM, 1988.

17. Kamiya, N. and Sawaki, Y., An efficient BEM for some inhomogeneous and nonlinear problems, in *Boundary Elements VII*, C. A. Brebbia and G. Maier, eds., Vol. II, Springer-Verlag, Berlin, 1985, 13.59–13.68.
18. Kantorovich, L. V. and Krylov, V. I., *Approximate Methods of Higher Analysis*, Wiley-Interscience, New York, 1958.
19. Lions, P. L., Méthode alterneé de Schwarz, manuscript.
20. Macaraeg, M. G. and Streett, C. L., Improvements in spectral collocation through a multiple domain technique, Inst. for Computer Applications in Science and Engineering, Hampton, VA, 1986.
21. Matsokin, A. M. and Nepomnyaschikh, S. V., On the convergence of the non-overlapping Schwarz subdomain alternating method, in *Met. Appr. in Interpolyatsii*, Yu. A. Kuznetsov, ed., Comp. Cent. Sib. Branch, USSR Acad. Sci., Novosibirsk, 1981, 85–97 (in Russian).
22. McKerrell, A., Phillips, C., and Delves, L. M., Chebyshev expansion methods for the solution of elliptic partial differential equations, *J. Comput. Phys.*, **37**, 1980, 444–452.
23. Morchoisne, Y., Inhomogeneous flow calculations for spectral methods, in *Spectral Methods for Partial Differential Equations*, R. G. Voigt, D. Gottlieb, and M. Y. Hussaini, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1984, 181–208.
24. Okamoto, N., Analysis of convective diffusion problem with first-order chemical reaction by boundary element method, *Intern. J. Numer. Meth. Fluids*, **8**, 1988, 55–64.
25. Okamoto, N., Boundary element method for chemical reaction system in convective diffusion, in *Numerical Methods in Laminar and Turbulent Flow*, C. Taylor, M. D. Olson, P. M. Gresho, and W. G. Habashi, Part I, 1985, 991–1002.
26. Orszag, S. A., Spectral methods for problems in complex geometries, *J. Comput. Phys.*, **37**, 1980, 70–92.
27. Patera, A. T., A spectral element method for fluid dynamics: Laminar flow in a channel expansion, *J. Comput. Phys.*, **54**, 1984, 468–488.
28. Rodrigue, G. and Saylor, P., Inner/outer iterative methods and numerical Schwarz algorithm II, *Proc. IBM Conf. on Vector and Parallel Processors for Scientific Computations*, Rome, Italy, 1985.
29. Rodrigue, G. and Simon, J., A generalization of the numerical Schwarz algorithms, in *Computing Methods in Applied Sciences and Engineering VI*, R. Glowinski and J. L. Lions, eds., North Holland, Amsterdam–New York, 1984.
30. Rodrigue, G., ed., *Parallel Processing for Scientific Computing*, SIAM, 1989.
31. Sakakihara, N., An iterative boundary integral method for mildly nonlinear elliptic partial differential equations, in *Boundary Elements VII*, C. A. Brebbia and G. Maier, eds., Vol. II, Springer-Verlag, Berlin, 1985, 13.49–13.58.
32. Schwarz, H. A., *Gesammelte Mathematische Abhandlungen*, Vol. 2, Springer-Verlag, Berlin, New York, 1980, 133–134.

33. Tsvik, L. B., Generalization of the Schwarz algorithm to the case of adjacent non-overlapping domains, *Doklady Akad. Nauk SSSR*, **224** (2), 1975, 309–312 (in Russian).
34. Widlund, O., Domain decomposition algorithms and the bicentennial of the French revolution, *SIAM News*, July 1989, 20.
35. Zanolli, P., Domain decomposition algorithms for spectral methods, *Calcolo*, to appear.

DISTRIBUTION LIST

	No. of Copies
Director Defense Tech. Information Center Cameron Station Alexandria, VA 22314	2
Director of Research Administration Code 012 Naval Postgraduate School Monterey, CA 93943	1
Library Code 0142 Naval Postgraduate School Monterey, CA 93943	2
Department of Mathematics Code MA Naval Postgraduate School Monterey, CA 93943	1
Center for Naval Analyses 4401 Ford Avenue Alexandria, VA 22302-0268	1
Professor Beny Neta Code MA/Nd Department of Mathematics Naval Postgraduate School Monterey, CA 93943	40
Professor Naotaka Okamoto Okayama University of Science Department of Applied Chemistry Ridai-cho 1-1, Okayama 700 Japan	20

Dr. C. P. Katti J. Nehru University School of Computer and Systems Sciences New Delhi 110067 India	1
Professor Paul Nelson Texas A&M University Department of Nuclear Engineering and Mathematics College Station, TX 77843-3133	1
Professor I. Michael Navon Florida State University Supercomputer Computations Research Institute Tallahassee, FL 32306	1
Professor M. M. Chawla, Head Department of Mathematics III/III/B-1, IIT Campus Hauz Khas, New Delhi 110016 India	1
Professor M. Kawahara Department of Civil Engineering Faculty of Science and Engineering Chuo University Kasuga 1-chome 13 Bunkyo-ku, Tokyo Japan	1

DUDLEY KNOX LIBRARY



3 2768 00347519 5